

A Majorization Order on Monomials and Termination of a Successive Difference Substitution Algorithm

Jia Xu

*College of Computer Science and Technology, Southwest University for Nationalities, Chengdu,
Sichuan 610041, PR China E-mail: j.jia.xu@gmail.com*

Yong Yao

*Chengdu Institute of Computer Applications, Chinese Academy of Sciences, Chengdu,
Sichuan 610041, PR China E-mail: yaoyong@casit.ac.cn*

Abstract When the ordering of variables is fixed, e.g., $x_1 \geq x_2 \geq \cdots \geq x_n$, the monomial $X^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ majorizing the monomial $X^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$ ($|\alpha| = |\beta|$) means that $\sum_{i=1}^k \alpha_i \geq \sum_{i=1}^k \beta_i$ ($k = 1, \dots, n-1$). In this paper, a necessary condition of positively terminating of a general successive difference substitution algorithm (KSDS) for an input f is obtained by using a majorization order on monomials. That is, every single term with negative coefficients in the form f is majorized at least by a single term with positive coefficients of f in an arbitrary ordering of variables.

Keywords Successive difference substitution algorithm, majorization order on monomials, termination, positive semi-definite form

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1 Introduction

The successive difference substitution algorithm based on A_n (SDS) is originated with a plain idea for proving homogeneous symmetric inequalities. It is firstly developed by L.Yang (see [1], [2], [3]) and after continuous improvement (see [4], [5]), now SDS has already come to be an effectual tool for solving many real algebra problems. Quite recently Y.Yao [5] established a new successive difference substitution algorithm based on G_n , which is named as NEWTSDS. NEWTSDS has many superior properties. Here A_n and G_n are respectively

$$A_n = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ & 1 & \ddots & \vdots \\ & & \ddots & 1 \\ 0 & & & 1 \end{pmatrix}, \quad G_n = \begin{pmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{n} \\ & \frac{1}{2} & \ddots & \vdots \\ & & \ddots & \frac{1}{n} \\ 0 & & & \frac{1}{n} \end{pmatrix}.$$

However, it is still very hard to fix a condition (necessary or sufficient) for termination of both SDS and NEWTSDS. In this paper, we will analyze the termination of a general successive difference substitution algorithm by using a majorization order on monomials. Finally, we have the following result.

Main result A necessary condition for positively terminating of the general successive difference substitution algorithm KSDS for an input f is that every single term with negative

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coefficients in a form f is majorized at least by a single term with positive coefficients of f in an arbitrary ordering of variables.

The structure of the paper is as follows. In section 2 we illustrate some background material of KSDS. In the following section, a main result of the majorization order on monomials is presented. Next, in section 4, conclusions are made and future research directions are outlined.

2 Background material of KSDS

Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, and $|\alpha| = \alpha_1 + \dots + \alpha_n$. A form (i.e., a homogeneous polynomial) f with degree d can be written as

$$f(x_1, \dots, x_n) = \sum_{|\alpha|=d} C_\alpha x_1^{\alpha_1} \dots x_n^{\alpha_n} = \sum_{|\alpha|=d} C_\alpha X^\alpha, \quad C_\alpha \in \mathbb{R}. \quad (1)$$

Definition 2.1^[5] A form f is defined to be trivially positive if the coefficient C_α of every single term X^α is nonnegative. If $f(1, 1, \dots, 1) < 0$ (i.e., the sum of coefficients of f is less than 0), then f is said to be trivially negative.

Definition 2.2 A form $f(X) \in \mathbb{R}[x_1, \dots, x_n]$ is positive semi-definite on \mathbb{R}_+^n if it satisfies $(\forall X \in \mathbb{R}_+^n) f(X) \geq 0$, where $\mathbb{R}_+^n = \{(x_1, \dots, x_n) | x_1 \geq 0, \dots, x_n \geq 0\}$. We note the set of all the positive semi-definite forms on \mathbb{R}_+^n as PSD. Furthermore, a form f is said to be positive definite on \mathbb{R}_+^n if $(\forall X \in \mathbb{R}_+^n, X \neq 0) f(X) > 0$. Correspondingly, the set of all the positive definite forms is briefly noted by PD.

Obviously, there is a trivial result reflecting the relationship between trivially positive (negative) and PSD: 1° If a form f is trivially positive, then $f \in \text{PSD}$. 2° If a form f is trivially negative, then $f \notin \text{PSD}$.

Next, let $q_1, \dots, q_n \in \mathbb{R}_+$ and

$$K_n = \begin{pmatrix} q_1 & q_2 & \cdots & q_n \\ & q_2 & \ddots & \vdots \\ & & \ddots & q_n \\ 0 & & & q_n \end{pmatrix}. \quad (2)$$

Suppose that S_n is a symmetric group of degree n , $\sigma \in S_n$, and P_σ is an $n \times n$ permutation matrix corresponding to σ . Thus we have the following definition.

Definition 2.3^[5] The $n \times n$ matrix B_σ is defined as

$$B_\sigma = P_\sigma K_n.$$

From Definition 2.3, we know that B_σ is obtained by permuting the rows of K_n . For example, given a permutation $\sigma = (1)(23) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$, then

$$P_{(1)(23)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad B_{(1)(23)} = P_{(1)(23)} K_3 = \begin{pmatrix} q_1 & q_2 & q_3 \\ 0 & 0 & q_3 \\ 0 & q_2 & q_3 \end{pmatrix}.$$

Definition 2.4^[5] Let $f \in \mathbb{R}[x_1, \dots, x_n]$, $X = (x_1, \dots, x_n)^T$, and S_n be a symmetric group of degree n . Define the set

$$\text{SDS}_K(f) = \bigcup_{\sigma \in S_n} f(B_\sigma X).$$

We call the set $\text{SDS}_K(f)$ as a set of difference substitution based on the matrix K_n .

It is easy to show the following equivalence relations (see [5])

$$f \in \text{PSD} \iff \text{SDS}_K(f) \subset \text{PSD}, \quad f \notin \text{PSD} \iff \exists g \in \text{SDS}_K(f), g \notin \text{PSD}.$$

Repeatedly using the above two equivalence relations and Definition 2.1, we have the following algorithm for testing positive semi-definite of polynomials, which is called as successive difference substitution algorithm based on the matrix K_n (KSDS).

Algorithm KSDS [5]

Input: A form $f \in \mathbb{Q}[x_1, x_2, \dots, x_n]$.

Output: " $f \in \text{PSD}$ ", or " $f \notin \text{PSD}$ ".

K1: Let $F = \{f\}$.

K2: Compute $T := \bigcup_{g \in F} \text{SDS}_K(g)$, Temp:= $T \setminus \{ \text{trivially positive polynomials of } T \}$.

K21: If Temp= \emptyset then return " $f \in \text{PSD}$ ".

K22: Else if there are trivially negative forms in Temp then return " $f \notin \text{PSD}$ ".

K23: Else let $F = \text{Temp}$ and go to step K2.

There is a fundamental question on the algorithm KSDS. That is, under what conditions does the algorithm terminates? This question is very hard to solve. Quite recently, Yang and Yao ([4], [5]) get some results for the termination of SDS and NEWTSDS.

Definition 2.5 The algorithm KSDS is positively terminating if the output is " $f \in \text{PSD}$ " when input f . The algorithm KSDS is negatively terminating if the output is " $f \notin \text{PSD}$ " when input f . Otherwise, KSDS is not terminating.

According to Definition 2.5, it is easy to get the following result.

Lemma 2.1 1. The algorithm KSDS is positively terminating for an input f iff, there is a positive integer m such that all of the coefficients of polynomial

$$f(B_{\sigma_1} B_{\sigma_2} \cdots B_{\sigma_m} X), \quad \forall \sigma_i \in S_n, \quad i = 1, \dots, m$$

are positive.

2. The algorithm KSDS is negatively terminating iff, there are m matrices $B_{\sigma_1}, B_{\sigma_2}, \dots, B_{\sigma_m}$ ($\sigma_i \in S_n$) such that

$$f(B_{\sigma_1} B_{\sigma_2} \cdots B_{\sigma_m} (1, 1, \dots, 1)^T) < 0.$$

In the next section, we will discuss necessary conditions of termination for KSDS by using a majorization order on monomials.

3 A majorization order on monomials and the main result

In general, we are not able to compare the following two monomials

$$X^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, X^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n} \quad (|\alpha| = |\beta|),$$

except in some added assumptions. For example, let $\alpha = (3, 1, 1), \beta = (2, 1, 2)$, and fix a order $x_1 \geq x_2 \geq x_3 \geq 0$, then we have

$$x_1^3 x_2 x_3 - x_1^2 x_2 x_3^2 = x_1^2 x_2 x_3 (x_1 - x_3) \geq 0.$$

This example inspired us to lead a majorization order on monomials into our analysis on the termination of KSDS.

Definition 3.1^{[6],[7]} Let $\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_n)$ ($\alpha, \beta \in \mathbb{R}_+^n, |\alpha| = |\beta|$). If

$$\sum_{i=1}^k \alpha_i \geq \sum_{i=1}^k \beta_i, \quad (k = 1, \dots, n-1) \quad (3)$$

then α is said to majorize β , and is denoted as $\alpha \succeq \beta$.

Definition 3.2 (Majorization Order) Let X^α, X^β ($|\alpha| = |\beta|$) be monomials, where the variables are followed in the ordering $x_1 \geq x_2 \geq \dots \geq x_n$. Suppose that σ is a permutation on the set $\{1, 2, \dots, n\}$. If $(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n)}) \succeq (\beta_{\sigma(1)}, \dots, \beta_{\sigma(n)})$ (briefly, $\alpha_\sigma \succeq \beta_\sigma$) then X^α is said to majorize X^β in the ordering of variables $x_{\sigma(1)} \geq x_{\sigma(2)} \geq \dots \geq x_{\sigma(n)}$, and is denoted as $(X^\alpha)_\sigma \succeq (X^\beta)_\sigma$.

The definition of majorization order on monomials is defined by making a comparison to the definition of majorization on symmetric polynomials (see [6], [7])

Note that $X^\alpha \equiv (X^\alpha)_\sigma \equiv X_\sigma^{\alpha_\sigma}$. Furthermore, there is

$$(X^\alpha)_\sigma \succeq (X^\beta)_\sigma \iff (X^{\alpha_\sigma})_I \succeq (X^{\beta_\sigma})_I,$$

where I is an identical permutation and can be omitted, e.g.

$$(x_1^3 x_2^4 x_3)_{(21)(3)} \succeq (x_1^4 x_2^2 x_3^2)_{(21)(3)} \iff x_2^4 x_1^3 x_3 \succeq x_2^2 x_1^4 x_3^2 \iff (4, 3, 1) \succeq (2, 4, 2).$$

It is easy to see that in the ordering $x_1 \geq x_2 \geq x_3$ the monomials $x_1^3 x_2^4 x_3$ and $x_1^4 x_2^2 x_3^2$ do not majorize each other. So the majorization order on monomials is a partial order and has the following three basic properties.

Lemma 3.1 Majorization order on monomials is reflexive, antisymmetric, and transitive, i.e., for a given permutation $\sigma \in S_n$, when $|\alpha| = |\beta| = |\gamma|$, we have that:

1. $(X^\alpha)_\sigma \succeq (X^\alpha)_\sigma$.
2. $(X^\alpha)_\sigma \succeq (X^\beta)_\sigma \wedge (X^\beta)_\sigma \succeq (X^\alpha)_\sigma \implies X^\alpha = X^\beta$.
3. $(X^\alpha)_\sigma \succeq (X^\beta)_\sigma \wedge (X^\beta)_\sigma \succeq (X^\gamma)_\sigma \implies (X^\alpha)_\sigma \succeq (X^\gamma)_\sigma$.

Lemma 3.2 Let $\sigma \in S_n$ be a given permutation. For the monomial X^α and X^β ($|\alpha| = |\beta|$), there is

$$(x_{\sigma(1)} \geq \dots \geq x_{\sigma(n)} \geq 0) \quad X^\alpha \geq X^\beta \iff (X^\alpha)_\sigma \succeq (X^\beta)_\sigma.$$

Lemma 3.3 Let $M = (p_{ij})_{n \times n}$ ($p_{ij} > 0$ if $i \leq j$, else $p_{ij} = 0$) be a upper triangular matrix. For $x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$, consider linear substitution $(x_1, \dots, x_n)^T = M(t_1, \dots, t_n)^T$

$$\begin{aligned} & (p_{11}t_1 + p_{12}t_2 + \dots + p_{1n}t_n)^{\alpha_1} (p_{22}t_2 + \dots + p_{2n}t_n)^{\alpha_2} \dots (p_{nn}t_n)^{\alpha_n} \\ &= \sum_{|(j_1, \dots, j_n)| = |\alpha|} C_{(j_1, \dots, j_n)} t_1^{j_1} t_2^{j_2} \dots t_n^{j_n}. \end{aligned}$$

Then the following results holds

$$C_{(j_1, \dots, j_n)} \neq 0 \iff (t_1^{\alpha_1} \dots t_n^{\alpha_n})_I \succeq (t_1^{j_1} \dots t_n^{j_n})_I.$$

The proof of the above three lemmas is very easy and can be omitted. Next we will focus on the proof of the main result.

Theorem 1 Suppose that f is a homogeneous polynomial of degree d in $\mathbb{Q}[x_1, \dots, x_n]$, which can be written as

$$f(x_1, \dots, x_n) = \sum_{|\alpha|=d} C_\alpha x_1^{\alpha_1} \dots x_n^{\alpha_n} = \sum_{|\alpha|=d} C_\alpha X^\alpha, \quad C_\alpha \neq 0. \quad (4)$$

For a monomial $C_\lambda X^\lambda$ in f , if it is not majorized by any other single terms in f under the ordering of variables $x_{\sigma(1)} \geq x_{\sigma(2)} \geq \dots \geq x_{\sigma(n)}$ ($\sigma \in S_n$), then the coefficient of the single term X^λ in $f(B_\sigma K_n^{m-1} X)$ is $(q_1^{\lambda_1} \dots q_n^{\lambda_n})^m C_\lambda$.

Proof According to (2), we know that K_n^m is a upper triangular matrix and the diagonal elements are q_1^m, \dots, q_n^m . Let

$$K_n^m = \begin{pmatrix} q_1^m & p_{12} & \cdots & p_{1n} \\ & q_2^m & \cdots & p_{2n} \\ & & \ddots & \vdots \\ 0 & & & q_n^m \end{pmatrix}, \quad (p_{ij} > 0, 1 \leq i < j \leq n).$$

and let

$$X' = \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix} = K_n^m X = \begin{pmatrix} q_1^m x_1 + p_{12} x_2 + \cdots + p_{1n} x_n \\ q_2^m x_2 + \cdots + p_{2n} x_n \\ \cdots \\ q_n^m x_n \end{pmatrix}.$$

Then by Definition 2.3 and (4) we have the following result.

$$\begin{aligned} f(B_\sigma K_n^{m-1} X) &= f(P_\sigma K_n^m X) = f(P_\sigma X') \\ &= f(x'_{\sigma(1)}, \dots, x'_{\sigma(n)}) = \sum_{|\alpha|=d} C_\alpha (X'^\alpha)_\sigma. \end{aligned}$$

Notice that the monomial X^λ is not majorized by any other single terms of f with variables being ordered as $x_{\sigma(1)} \geq x_{\sigma(2)} \geq \dots \geq x_{\sigma(n)}$. So by Lemma 3.1 and Lemma 3.3, the monomial $(X^\lambda)_\sigma$ in $f(B_\sigma K_n^{m-1} X)$ is only generated by expanding $(X'^\lambda)_\sigma$. Note that

$$\begin{aligned} (X'^\lambda)_\sigma &\equiv X'^\lambda \\ &\equiv (q_1^m x_1 + p_{12} x_2 + \cdots + p_{1n} x_n)^{\lambda_1} (q_2^m x_2 + \cdots + p_{2n} x_n)^{\lambda_2} \cdots (q_n^m x_n)^{\lambda_n}. \end{aligned}$$

Thus the coefficient of X^λ is $(q_1^{\lambda_1} \dots q_n^{\lambda_n})^m C_\lambda$. \blacksquare

By Lemma 2.1 and Theorem 1, we immediately have the following main result.

main result A necessary condition for positively terminating of KSDS for an input f is that every single term with negative coefficients in a form f is majorized at least by a single term with positive coefficients of f in an arbitrary ordering of variables.

For example, let us consider the cyclic polynomial

$$f = x_1^4 x_2^2 - x_1^3 x_2 x_3^2 + x_2^4 x_3^2 - x_1^2 x_2^3 x_3 + x_1^2 x_3^4 - x_1 x_2^2 x_3^3.$$

Note that the single term $x_1^3 x_2 x_3^2$ with negative coefficients is not majorized by the single terms $x_1^4 x_2^2$, $x_2^4 x_3^2$, $x_1^2 x_3^4$ with positive coefficients in the ordering $x_1 \geq x_3 \geq x_2$. Choose the following matrix A_3 , and let the permutation $\sigma = (1)(23)$.

$$A_3 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_{(1)(23)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

By Theorem 1, after the polynomial $f(P_{(1)(23)} A_3^m X)$ being expanded the coefficient of $x_1^3 x_2 x_3^2$ is always -1. So SDS (based on A_3) is not positively terminating for an polynomial

f inputed. (by other ways we can prove that $(\forall X \in \mathbb{R}_+^3)f \geq 0$, so SDS is not negatively terminating yet.)

From the other point of view, we can compute $P_{(1)(23)}A_3^m$ by using Jordan normal form:

$$P_{(1)(23)}A_3^m = \begin{pmatrix} 1 & m & m(m-1)/2 \\ 0 & 0 & 1 \\ 0 & 1 & m \end{pmatrix}.$$

The coefficient of $x_1^3x_2x_3^2$ is still -1 by expanding $f(P_{(1)(23)}A_3^mX)$. Thus results got from the above two methods are congruent.

4 Conclusion

There are many interesting questions on successive difference substitution algorithms, e.g., how to specify the necessary and sufficient conditions for positive termination of the algorithm KSDS, and how to define the necessary and sufficient conditions for negative termination of it? We can put the work on it in the following two aspects.

1. Yang and Yao ([4],[5]) have already proved that the necessary and sufficient condition for negative termination of SDS and NEWTSDS is $f \notin \text{PSD}$. So for KSDS we give the following conjecture.

Conjecture The algorithm KSDS is negatively terminating iff $f \notin \text{PSD}$.

2. However it is more difficult to study positive termination of KSDS. For positive termination of NEWTSDS, Yao ([5]) has proved the following result.

Theorem 2^[5] Let $f(X) \in \mathbb{R}[x_1, \dots, x_n]$. If $(\forall X \in \mathbb{R}_+^n, X \neq 0) f(X) > 0$, then there exists $m > 0$ such that the coefficients of

$$f(B_{\sigma_1}B_{\sigma_2} \cdots B_{\sigma_m}X), \forall \sigma_i \in S_n (B_{\sigma_i} = P_{\sigma_i}G_n)$$

are all positive.

Theorem 2 indicates that NEWTSDS is positively terminating for a form in PD.

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